Motivation

Recall that. let E = cpx-ori. cohomology thy $\rightsquigarrow$ spectrum E.

$E^*(\mathbb{CP}^\infty) = E^*[\![x]\!] \longrightarrow E^*[\![x, y]\!] = E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \rightsquigarrow$ f.g.l.

f.g.l. (assemble into formal gps) F / R can be written as

$$[n]_F\, x = F(x, [n-1]_F\, x) = x +_F [n-1]_F\, x.$$

$$[0]_F\, x = 0$$

Take $n = p$ prime. we have $[p]_F\, x$. If $\exists\ v_i \in R$ s.t.

$$[p]_F\, x = px +_F v_1 x^p +_F \cdots +_F v_n x^{p^n} +_F \cdots$$

Then F is p-typical. Special case is the Honda f.g.l. Γ_n over $\mathbb{F}_{p^n}$ w/ $[p]_{\Gamma_n}\, x = x^{p^n}$. Another special example is the Lubin-Tate f.g.l. which p-series is given by $[p]_{LT}\, x = px +_F x^{p^n}$.

Note that for $F \in \mathrm{Alg}_{\mathbb{F}_p}$. F has ht n if $[p]_F\, x = g(x^{p^n})$. $g'(0) = 0$

ht $F_a = \infty$. ht $F_m = 1$.

Note that the Lubin-Tate theory concerns w/ the universal deformation.

Namely. $\forall$ 1st order deformation A of k, the f.g.l. F over R,

$R = W_p(k)[\![v_1, \dots, v_{n-1}]\!]$. $n = $ ht F. induces an bijection

$$\mathrm{Hom}_k(R, A) \longrightarrow \mathrm{Def}(A).$$

This f.g.l. implies $\exists$ ring homomphism $MU_{(p)*} \cong L_{(p)} \longrightarrow R$. and in fact $v_0 = p, v_1, v_2, \dots, v_{n-1}$ are regular (i.e. $v_n : R/I_{p,n} \longrightarrow R/I_{p,n}$ inj. $I_{p,n} = (p, v_1, \dots, v_{n-1})$ and $v_i = a_{p^i - 1}$ for $[p]_{MU_*}\, x = \sum_{k=0}^{\infty} a_k x^{k+1}$).

$\Rightarrow$ F is Landweber exact. By LEFT. $X \longmapsto MU^*(X) \otimes_{MU_*} R$

gives a cohomology thy rep. by $E(n)$. which is even periodic

$$\pi_* E(n) = E(n)^*(*) = R[\beta^{\pm}]$$

The Morava stabilization gp $\mathbb{G}$ acts on $\pi_* E(n)$. Now $\mathbb{G}$ is defined to be $\mathrm{Aut}(F_n, \overline{\mathbb{F}_p})$. F_n = f.g.l. of ht n on $\overline{\mathbb{F}_n}$. and $\mathcal{M}_{fg}^{=n} = \mathrm{Spec}\, \overline{\mathbb{F}_p} / \mathbb{G}$ gives a stratification on $\mathcal{M}_{fg}$.
(moduli stack of formal gps: $\mathcal{M}_{fg}: \mathrm{CRing} \longrightarrow \mathrm{Gpd}$. $R \mapsto (FG(R)$. isos$)$). It satisfies

$$0 \to \mathrm{Aut}\, F_n \to \mathbb{G} \to \mathrm{Gal}(\overline{\mathbb{F}_p}, \mathbb{F}_p) \to 0$$

Problems

1) Can the action of $\mathbb{G}$ on $\pi_* E(n)$ be lifted to ones on $E(n)$?

2) Does $E(n)$ possess some nice structure (e.g. A_∞, E_∞)?

3) Any other thing you can say about the functor $(k, F) \mapsto E(k, F)$?

Thm (Goerss - Hopkins) $B\mathbb{G} \simeq$ moduli space of E_∞-ring that is equiv to Lubin-Tate theory $E(k, F)$ as htpy assoc. rings.

Essentially, Goerss - Hopkins: functor: f.g.l / $k \longrightarrow E_\infty$-ring.
Hopkins - Miller: functor: $\cdot\cdot \longrightarrow A_\infty$-ring.

Goerss - Hopkins Obstruction Theory

$\rightsquigarrow$ Try to solve : When $\exists$ E_∞-ring R s.t. $E_*R \cong A$ in E_*E-comodule algs. given A E_*E-comodule alg. E_*E flat / E_*, E htpy comm. ring spectrum?

▶ Modern approach (refer to : abstract Goerss-Hopkins obstruction by Pstragowski and Van Koughnett. 2019) uses synthetic spectra. I know nothing about them !

Recall Barratt - Eccles operad $\mathcal{E}$ is an example of E_∞-operad.

$\mathcal{E}(n) = E\Sigma_n$ universal principle bundle for Σ_n

$$\gamma : \mathcal{E}(k) \times \mathcal{E}(j_1) \times \cdots \times \mathcal{E}(j_k) \longrightarrow \mathcal{E}(j)$$

induced by $\Sigma_k \times \Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \longrightarrow \Sigma_j$.

Given $X \in Sp$. free E_∞-ring spectrum $\mathcal{E}(X)$ has htpy type of

$$\bigvee_{n \geq 0} (E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n}$$

- Issues :

① To compute $E_*\mathcal{E}(X)$, need to know $E_*B\Sigma_n$ $\Rightarrow$ need to know E_* - Dyer - Lashof operations.

② Want to realize A in E_*-alg over E_*E-comodule as E_∞-ring we may not want a Dyer - Lashof alg str. on A since A is only assumed to be "no more than" commutative.

- Solution: resolve E_∞-operad by simplicial operad that yields the desired flexibility and possibility of computing E_*-homology of free object

  $\longrightarrow$ This is where the André-Quillen homology get involved.

- Strategy: "moduli approach". Let $\mathcal{F} \in \mathcal{O}$ = cat of operads in sSet.

  $A = E_*\mathcal{F}$-alg in E_*E-comodule. Both $\mathcal{F}$ & A assoc. / comm.

  $\mathcal{E}(A)$ = cat of $\mathcal{F}$-alg sp R s.t. $E_*R \cong A$ as $E_*\mathcal{F}$-algs.

  mor = E_*-iso.

  Define $TM(A) := B\mathcal{E}(A)$. To study whether this is nonempty.

Rk. We need E to satisfy the Adams condition to ensure that the (co)homology over E has Künneth SS:

$$E^2_{p,q} = \bigoplus_{k_1+k_2=q} \mathrm{Tor}_p^{E_*}(E_*(X), E_*(Y)) \Rightarrow E_*(X \wedge Y).$$

- Adams conditions

  $E \overset{\text{w.e.}}{\simeq} \operatorname{hocolim} E_\alpha$. E_α finite cellular s.t.

  1) $E_* DE_\alpha$ proj E_*-module

  2) $\forall M$ E-module spectrum. $[DE_\alpha, M] \longrightarrow \mathrm{Hom}_{E_*}(E_*(DE_\alpha), M)$ iso.

- e.g. $\mathbb{S}$, $H\mathbb{F}_p$, MO, MU, any Landweber exact theory, in particular Lubin-Tate theory.

  non-e.g. $H\mathbb{Z}$.

Step 1 $TM(A)$ hard, but $TM_n(A)$ is NOT !

First resolve $\mathcal{F}$ that yields the flexibility, and possibility of computing E_*-homology of free obj.

Thm $\exists$ simplicial operad $T \to \mathcal{F}$ s.t.

1) T Reedy cofibrant (i.e. levelwise cofibrant)

2) $\forall\, n \geq 0,\ q \geq 0$. $\pi_0 T_n(q)$ is a free Σ_q-set

3) $|T| \to \mathcal{F}$ by augmentation is w.e.

4) If $E_*\mathcal{F}(q)$ proj E_*-module, $\forall q$, then

E_*T is cofibrant as simplicial operad in E_*-module

and $E_*T \to E_*\mathcal{F}$ w.e. in this cat.

▲ Technical Part : model str on sSp can be lifted to the cat $sAlg_T$.

Sketch : Let $\mathcal{P}$ = minimal collection of spectra s.t.

1) $S \in \mathcal{P}$

2) $DE_\alpha \in \mathcal{P}$

3) $\mathcal{P}$ closed under Σ, Σ^{-1}, $\vee$

4) $\forall\, P \in \mathcal{P}$, M E-module, $[P, M] \to \mathrm{Hom}_{E_*}(E_*P, M_*)$

Use these bricks to build cofibrant resolution & replacement.

$\mathcal{P}$-w.e. : $\pi_*[P, X_\bullet] \to \pi_*[P, Y_\bullet]$ iso, $\forall P$

$\mathcal{P}$-fib : $\quad$ " $\quad$ " $\quad$ fib, $\forall P$.

$\rightsquigarrow$ model str on sSp.

To lift to model str on $sAlg_T$. use the SS

$$\pi_{s,t}(X_\cdot\,;P) = \pi_s[\Sigma^t P,\ X_\cdot] \Rightarrow [\Sigma^{s+t}P,\ |X_\cdot|]$$

and map of SSs

$$\begin{tikzcd}
\pi_{s,t}(X_\cdot\,;P) \arrow[r, Rightarrow] \arrow[d] & {[\Sigma^{s+t}P,\ |X_\cdot|]} \arrow[d] \\
\pi_{s,t}(Y_\cdot\,;P) \arrow[r, Rightarrow] & {[\Sigma^{s+t}P,\ |Y_\cdot|]}
\end{tikzcd}$$

P-w.e. $X_\cdot \to Y_\cdot$ gives rise to iso of E_2-pages $\rightsquigarrow$ w.e. in $sAlg_T$.

similar for others.

Def $X \in sAlg_T$ is potential n-stage if $(0 \le n \le \infty)$

$$\pi_i E_* X = \begin{cases} A, & i = 0 \\ 0, & 1 \le i \le n+1 \\ \text{whatever not too ridiculous}, & i \ge n+2 \end{cases}$$

$\pi_i^{\natural}(X;P) = [P \wedge \Delta^n/\partial\Delta^n,\ X]_\varphi = 0$, $i > n$

$TM_n(A)$ = moduli space of all $X \in sAlg_T$ which are potential n-stage

$\exists\ P_m : TM_n(A) \longrightarrow TM_m(A)$, $0 \le m \le n \le \infty$.

Th 1 1-1: $TM_\infty(A) \longrightarrow TM(A)$ w.e., and

$TM_\infty(A) \longrightarrow \operatorname{holim} TM_n(A)$ w.e.

Now there's a tower:

$$
\begin{array}{c}
TM_\infty(A) \\
\downarrow \\
\vdots \\
\downarrow \\
TM_n(A) \\
\downarrow \\
TM_{n-1}(A) \\
\downarrow \\
\vdots \\
\downarrow \\
TM_1(A) \\
\downarrow \\
TM_0(A)
\end{array}
$$

<u>Step 2</u> Get info of $TM_0(A)$ and get n-stage from $(n-1)$-stage. then the result follows.

- For $TM_0(A)$:

  Thm $\mathrm{Aut}\,A$ = automorphism of E_*F-alg A in E_*E-comodules w/ discrete top.

  Then $TM_0(A) \simeq B\mathrm{Aut}(A)$ w.e.

- From TM_{n-1} to TM_n: $n \geq 1$

Th $\exists$ htpy pullback

$$\begin{tikzcd} TM_n(A) \arrow[r] \arrow[d] & B\,\mathrm{Aut}(A, \Omega^n A) \arrow[d, "\varphi"] \\ TM_{n-1}(A) \arrow[r] & \widehat{\mathcal{H}}^{n+2}(A, \Omega^n A) \end{tikzcd}$$

$\mathrm{fib}\,\varphi = \widehat{\mathcal{H}}^{n+1}(A, \Omega^n A)$. where $\widehat{\mathcal{H}}^{n+2}(A, \Omega^n A)$ is a space s.t.

$\pi_0 \widehat{\mathcal{H}}^{n+2}(A, \Omega^n A) = D^{n+2}_{E_*T/E_*E}(A, \Omega^n A)$. where

D = André - Quillen cohomology.

Cor Obstructions to existence of realization of A by $\mathcal{F}$-alg live in $D^{n+2}_{E_*T/E_*E}(A, \Omega^n A)$.

Obstructions to uniqueness .. .. live in $D^{n+1}_{E_*T/E_*E}(A, \Omega^n A)$

Application to Lubin - Tate

We've been told that Lubin - Tate theory $E(k, F)$ satisfies the Adams condition. Let $E = E(k_1, F_1)$. Here's the corollary of Goerss - Hopkin obstruction theory:

Cor $TM(E_*E) \simeq B\,\mathrm{Aut}(k_1, F_1) = B\mathbb{G}$. In particular, E has a unique E_∞ - str realizing E_*E as a comm. E_* - alg in E_*E - comodules.

Sketch of pf.

1) The obstructions to existence & uniqueness live in

$D^*_{E_*T/E_*E}(E_*E, \Omega^n E_*E)$, $\forall n$.

So it suffices to show this is 0. To simplify this, use the fact:

<u>FACT</u> $D^*_{C/E_*E}(-, E_*E \otimes_{E_*} M) \cong D^*_C(-, M)$

for C = simplicial operad in E_*E-comodule

M = A-module in E_*-module

A = $\pi_0 C$-alg in E_*E-comodule.

$\rightsquigarrow$

$$D^*_{E_*T/E_*E}(E_*E, \Omega^n E_*E) \cong D^*_{E_*T}(E_*E, \Omega^n E_*).$$

Recall that AQ homology is

$$D^*(A, M) = H^*(\mathrm{Hom}_A(L_{A/k}, M))$$

So $D^*_{E_*T}(E_*E, \Omega^n E_*) = D^*_{E_*T}(E_*E, E_{*+n})$

$$= \mathrm{Ext}^{*,*}_{E_*}(L_{E_*E/E_*}, E_*)$$

Filter E_* by powers of its maximal ideal $m = (p, v_1, \dots, v_{n-1})$

$\rightarrow$ SS computing $\mathrm{Ext}^{*,*}_{E_*}(L_{E_*E/E_*}, E_*)$ w/ E_2-term

$$\mathrm{Ext}^{p,*}_{E_*/m}(L_{E_*E/E_*} \otimes^{\mathbb{L}}_{E_*} E_*/m, m^q/m^{q+1})$$

Suffice to show

$$L_{E_*E/E_*} \otimes^{\mathbb{L}}_{E_*} E_*/m \simeq L_{(E_*E/m)/(E_*/m)}$$

$$\simeq E_* \otimes_{E_0} L_{(E_0E/m)/(E_0/m)}$$

$$\simeq 0 \quad \text{by flat base change}$$

$$(E_*E \text{ flat})$$

Now since $E_0/m \cong k$ perfect, $E_0E/m \cong \mathrm{Hom}_{cts}(\mathbb{G}, k)$

FACT cotangent cpx of any mphism between perfect $\mathbb{F}_p$-algs vanishes.

pf. σ = Frob endomphism. $\sigma: E_0E/m \to E_0E/m$

It is actually an automphism since k perfect.

$L_{(E_0E/m)/(E_0/m)} = \Omega_{Q_\bullet/(E_0/m)} \otimes_{Q_\bullet} E_0E/m$

for $Q_\bullet \to E_0E/m$ cofib simplicial resolution.

But now

$$\sigma^*: L_{(E_0E/m)/(E_0/m)} \to L_{(E_0E/m)/(E_0/m)}$$

$\sigma^*(dx) = d\sigma^*(x) = dx^p = px^{p-1}dx = 0$ since working in

char p.

Hence obstructions vanish $\Rightarrow$ $TM(E_*E) \simeq B\mathbb{G}$.

In particular, E has E_∞-ring str.

2) By the fact

FACT $\mathrm{Hom}_{E_*}(E_*F, A_*) \cong$ iso in FGL_n $(k_1, F_1) \to (A_0/m, \iota^*F_2)$

where $\iota: E_* \to A_*$ cts at deg 0. $F = E(k_2, F_2)$, m max

ideal in A_0.

It follows that $\mathrm{Aut}(E_*E) \cong \mathrm{Aut}(k, F) = \mathbb{G}$.